

# STAR OPERATION ON ORDERS IN SIMPLE ARTINIAN RINGS

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ABSTRACT. Star operations are an important tool in multiplicative ideal theory. In this paper we apply a special type of star operation, known as  $\nu$ -operation, to define the notion of right Prüfer  $\nu$ -multiplication order. The latter may be viewed as a natural non-commutative version of Prüfer  $\nu$ -multiplication domain. As one of our main results, we establish that an overring of a right Prüfer  $\nu$ -multiplication order is again a right Prüfer  $\nu$ -multiplication order.

## 1. INTRODUCTION

Multiplicative ideal theory is a crucial ingredient in the classification of orders in simple Artinian rings [18, 23, 24]. In turn, star operations are a powerful tool used to study multiplicative ideal theory. Most progress, however, is concerned with the application of star operations in the commutative setting, see [2, 11, 12, 27] and references therein, and relatively little is known in the non-commutative case. To the best of our knowledge, the first to advance the latter were Asano and Murada [4], who, in 1953, used the  $\nu$ -operation to study the Arithmetic properties of non-commutative semigroups. Surprisingly, only a handful of further works have been devoted to the application of star operations in non-commutative ring theory. For example, in [5, 7] the  $\nu$ -operation is used to classify prime segments of Dubrovin valuation rings, and right cones in right orderable groups. Furthermore, in [23], Marubayashi applied the  $\nu$ -operation to investigate Ore extensions over total valuation rings. He defined the notion of  $\nu$ -Bezout orders, and proved that an order in a simple Artinian ring is  $\nu$ -Bezout if and only if it is a GCD order.

In this article we build on Marubayashi's work, to further our understanding of  $\nu$ -multiplication orders. We study both the  $\nu$ -Bezout order introduced in [23] as well as a new order, called right Prüfer  $\nu$ -multiplication order, which is a non-commutative version of Prüfer  $\nu$ -multiplication domain. For this new order we prove that an overring of a right Prüfer  $\nu$ -multiplication order is again a right Prüfer  $\nu$ -multiplication order.

The remainder of the paper organised as follows. In the next section our main theme of study is the  $\nu$ -Bezout order. In particular, in Theorem 2.3, we show that for total valuation rings  $V \subset W$  of a division ring  $K$ , the  $\nu$ -Bezout order  $V + W[x, \sigma]x$  in  $K(x, \sigma)$  is Bezout if and only if  $W = K$ . In Section 3, our focus is the  $\tau$ -operation, which, among other things, is needed in our subsequent study of right Prüfer  $\nu$ -multiplication order in Section 4. In that particular section, we first define this new order and then prove that an overring of a right Prüfer  $\nu$ -multiplication is itself a right Prüfer  $\nu$ -multiplication order.

2.  $\nu$ -BEZOUT ORDERS

Let  $S$  be an order in a simple Artinian ring  $Q$ , i.e.,  $S$  is a prime Goldie ring with total quotient ring  $Q$ . Given regular elements  $a, b$  in  $S$ , we say that  $b$  is a *right-divisor* of  $a$ , if there exists an element  $s \in S$  such that  $a = bs$ . A *left-divisor* is defined similarly.

**Definition 2.1.** Let  $a, b, d \in S$  be regular elements of  $S$ . We say that  $d$  is a *right greatest common divisor* of  $a$  and  $b$ , denoted by  $d = r - \gcd\{a, b\}$ , if the following two conditions hold:

- (i)  $d$  is a right-divisor of  $a$  and  $b$ ;
- (ii) if  $c$  is a right-divisor of  $a$  and  $b$ , then  $c$  is a right-divisor of  $d$ .

Note that if  $c$  is another right greatest common divisor of  $a$  and  $b$ , then  $dS = cS$ .

A left greatest common divisor of  $a, b$ , denoted by  $l - \gcd\{a, b\}$ , is defined likewise. An order  $S$  in a simple Artinian ring  $Q$  is called a GCD order if any two elements of  $S$  have a right as well as a left greatest common divisor.

Let  $U(Q)$  is the group of units in  $Q$ . A right  $S$ -submodule  $I$  of  $Q$  is called a *right  $S$ -ideal* if  $I$  contains a regular element in  $S$  and  $uI \subseteq S$  for some  $u \in U(Q)$ . For any subsets  $A$  and  $B$  of  $Q$ , we use the notations:  $(A : B)_r = \{q \in Q : qB \subseteq A\}$  and  $(A : B)_l = \{q \in Q : Bq \subseteq A\}$ . If  $I$  is a right  $S$ -ideal, then  $(S : I)_l$  is a left  $S$ -ideal. We define  $I_\nu := (S : (S : I)_l)_r$ . The set  $I_\nu$  is a right  $S$ -ideal containing  $I$ . If  $I_\nu = I$ , then it is called a *right  $\nu$ -ideal*. Similarly, for any left  $S$ -ideal  $J$ , we can define a left  $S$ -ideal  ${}_\nu J$ . An order  $S$  in  $Q$  is called a *right  $\nu$ -Bezout order* if  $I_\nu$  is right principal for any finitely generated right integral  $S$ -ideal  $I$ . A *left  $\nu$ -Bezout order* is defined similarly. By a  *$\nu$ -Bezout order* we mean it is a right as well as left  $\nu$ -Bezout order. In [23, Proposition 2.2] Marubayashi proved that an order  $S$  in a simple Artinian ring  $Q$  is a  $\nu$ -Bezout order if and only if  $S$  is a GCD order.

**Lemma 2.2.** Let  $V$  be a total valuation ring of a division ring  $K$  and let  $I$  be an ideal of  $T := V + K[x, \sigma]x$ . Then the following three statements are equivalent:

- (1)  $I \cap V \neq 0$ ;
- (2)  $K[x, \sigma]x \subset I$ ;
- (3)  $IK[x, \sigma] = K[x, \sigma]$ .

If any of the above conditions hold, then  $I = (I \cap V) + K[x, \sigma]x = (I \cap V)T$ .

*Proof.* The proof is similar to the commutative case, see [9, Lemma 4.11]. □

Let  $V$  be a proper total valuation ring of a division ring  $K$  and  $\sigma$  be an automorphism of  $V$ . Let  $V[x, \sigma]$  be the ring of Ore extension over  $V$  with multiplication  $xa = \sigma(a)x$  for all  $a \in V$ . The automorphism  $\sigma$  thus naturally extends to an automorphism of  $K$ . Then  $K[x, \sigma]$  is a principal ideal ring and has the division ring  $K(x, \sigma)$  as quotient ring. In [23] it is proved that  $V[x, \sigma]$  is a  $\nu$ -Bezout order in  $K(x, \sigma)$ , which is not Bezout. The following is a similar type of result but with a different proof strategy.

**Theorem 2.3.** Let  $V$  be a total valuation ring of a division ring  $K$  and let  $W$  be a proper overring of  $V$ . Then the following three statements are equivalent.

- (1)  $V + W[x, \sigma]x$  is a  $\nu$ -Bezout order in  $K(x, \sigma)$ ;
- (2)  $W = K$ ;
- (3)  $V + W[x, \sigma]x$  is a Bezout order in  $K(x, \sigma)$ .

*Proof.* **(1)  $\Rightarrow$  (2).** Since  $W$  is a proper overring of the total valuation ring  $V$ , we have  $W = V_S$ , where  $S = V - J(W)$  and  $V_S$  is the localization of  $V$  at the Ore system  $S$ . Therefore,  $V + W[x, \sigma]x = V + V_S[x, \sigma]x$ . Put  $V^{(S)} := V + V_S[x, \sigma]x$ . To prove that  $W = K$ , it is enough to show that  $J(W) = 0$ . Proceeding by contradiction, assume there exists a nonzero element  $a \in J(W)$ . Then  $aV^{(S)} \subseteq sV^{(S)}$  for all  $s \in S$ . Since  $x = ss^{-1}x$ , we also have  $xV^{(S)} \subseteq sV^{(S)}$  for all  $s \in S$ . Now let  $d = l - \gcd\{a, x\}$ . Then  $a = df$  and  $x = dg$  for some  $f, g \in V^{(S)}$ . From  $x = dg$  and the fact that  $d$  is a constant, we conclude that  $d \notin J(W)$ , which implies that  $d \in S$ . Therefore,  $xV^{(S)} \subseteq d^n V^{(S)}$  and  $aV^{(S)} \subseteq d^n V^{(S)}$  for all  $n \geq 0$ . Now if  $d$  is a non-unit in  $V$ , then this is in contradiction with  $\gcd\{d^{-1}a, d^{-1}x\} \in U(V^{(S)})$ . If  $d$  is a unit in  $V$ , then this is in contradiction with  $V \neq W$ . Therefore our assumption on the existence of  $a$  is false, and  $J(W) = 0$ . This concludes the proof that (1) implies  $W = K$ .

**(2)  $\Rightarrow$  (3).** Let  $T = V + K[x, \sigma]x$ . First we will show that each right ideal of  $T$  is of the form  $f(x)FT = f(x)(F + K[x, \sigma]x)$ . To see this, let  $I$  be a right ideal of  $T$ . If  $IK[x, \sigma] = K[x, \sigma]$ , then by Lemma 2.2,  $I \cap V \neq 0$  and  $I = I \cap V + K[x, \sigma]x = (I \cap V)T$ . Thus it is enough to consider  $f(x) = 1$  and  $F = I \cap V$ .

Since  $K[x, \sigma]$  is principal ideal ring, if  $IK[x, \sigma] \neq K[x, \sigma]$  then  $IK[x, \sigma] = f(x)K[x, \sigma]$  for some  $f(x) \in K[x, \sigma]$ . Hence there exist  $a_i \in I$  and  $h_i \in K[x, \sigma]$  such that  $f(x) = \sum_{i=0}^n a_i h_i$ . Let  $h_i = \sum_j q_{i,j} x^j$ . Then  $f(x) = \sum_{i=0}^n a_i q_{i,0} + \sum_{i=0}^n a_i h'_i$ , where  $q_{i,0} \in K$  and  $h'_i \in K[x, \sigma]x$ . Since  $V$  is a total valuation ring of  $K$ , there exist  $t \in V$  such that  $q_{i,0}t \in V$  for all  $1 \leq i \leq n$ . Hence  $f(x)t = \sum_{i=0}^n (a_i q_{i,0}t) + \sum_{i=0}^n (a_i h'_i t)$ . Since  $a_i \in I$ ,  $q_{i,0}t \in V$  and  $h'_i t \in K[x, \sigma]x$ , we have  $f(x)t \in I$ . Let

$$(2.1) \quad F = \{t \in V : f(x)t \in I\}.$$

Then  $F$  is a nonzero right  $V$ -submodule  $K$ . From  $f(x)F \subseteq I$  and  $FT = F + K[x, \sigma]x$ , we have  $I \supseteq f(x)FT = f(x)(F + K[x, \sigma]x)$ . Conversely, let  $h(x) \in I$ . Then  $h(x) = f(x)(q_0 + q_1x + \cdots + q_mx^m)$ , where  $q_i \in K$ . Put  $h'(x) = f(x)(q_1x + \cdots + q_mx^m)$ . Then  $h'(x) \in f(x)K[x, \sigma]x$  and  $h(x) = f(x)q_0 + h'(x)$ . Since  $f(x)(q_1x + \cdots + q_mx^m) \subseteq I$ , we have  $f(x)q_0 = h(x) - h'(x) \in I$ . Thus  $q_0 \in F$  and  $h(x) \in f(x)(F + K[x, \sigma]x)$  so that  $I \subseteq f(x)(F + K[x, \sigma]x)$ . Therefore,  $I \supseteq f(x)FT \supseteq f(x)(F + K[x, \sigma]x) \supseteq I$ , which shows that  $I = f(x)FT = f(x)(F + K[x, \sigma]x)$ .

Now let  $I$  be a finitely generated right ideal of  $T$ . Then  $F$  defined in (2.1) is a finitely generated right  $V$ -module such that  $I = f(x)FT$ . By construction  $F = d_1V + \cdots + d_nV$ , where  $d_1, \dots, d_n \in V$ . Therefore,  $F = dV$  for some  $d \in V$  and  $I = f(x)dVT = f(x)dT$ , which shows that  $T$  is a right Bezout ring. Similarly one can prove that  $T$  is a left Bezout ring.

**(3)  $\Rightarrow$  (1).** A Bezout order is always a  $\nu$ -Bezout order.  $\square$

**Lemma 2.4.** *Let  $S$  be a  $\nu$ -Bezout order in a division ring. For any triple of regular elements  $a, b, c \in R$ , if  $\gcd\{a, b\} = 1$  then  $\gcd\{a, bc\} = \gcd\{a, c\}$ .*

*Proof.* Suppose  $d = l - \gcd\{a, c\}$  and  $a = a'd, c = c'd$ . Then by [23, Lemma 2.1],  $l - \gcd\{a', c'\} = 1$ . From  $l - \gcd\{a', c'\} = 1$  and  $l - \gcd\{a', b\} = 1$  we conclude that  $l - \gcd\{a', bc'\} = 1$ . Again by [23, Lemma 2.1],  $l - \gcd\{a, bc\} = d$ , as desired. Likewise one can prove that  $r - \gcd\{a, bc\} = r - \gcd\{a, c\}$ .  $\square$

**Corollary 2.5.** *For any  $a, b, c \in S$ , if  $\gcd\{a, b\} = 1$  and  $a$  is a divisor of  $bc$ , then  $a$  is a divisor of  $c$ .*

**Definition 2.6.** Let  $S$  be a  $\nu$ -Bezout order in a division ring  $D$ . A prime ideal  $P$  of  $S$  is a left PF-prime ideal if  $l - \gcd\{a, b\} \in P$  for any pair of regular elements  $a, b \in P$ .

**Theorem 2.7.** Suppose that  $P$  is a completely prime ideal of a  $\nu$ -Bezout order  $S$  such that  $S$  is localizable at  $P$ . Then  $P$  is PF-prime if and only if  $S_P$  is a total valuation ring of  $D$ .

*Proof.* Let  $P$  be a left PF-prime ideal of  $S$  and  $x = ab^{-1} \in D$ , where  $a, b \in S$  such that  $l - \gcd\{a, b\} = 1$ . Since  $1 \notin P$ , we have  $a \notin P$  or  $b \notin P$ . Thus  $x = ab^{-1} \in S_P$  or  $x^{-1} = ba^{-1} \in S_P$ .

Conversely, suppose  $P$  is not a left PF-prime. Then there exist  $a, b \in P - \{0\}$  such that  $l - \gcd\{a, b\} \notin P$ . Let  $d = l - \gcd\{a, b\}$  and  $x = ad^{-1}, y = bd^{-1}$ . Then  $x, y \in P$  and, by [23, Lemma 2.1],  $l - \gcd\{x, y\} = 1$ . We will show that neither  $xy^{-1}$  nor  $yx^{-1} \notin S_P$ . If  $xy^{-1} \in S_P$  then  $xy^{-1} = ts^{-1}$  for some  $t \in S$  and  $s \in S - P$ . Since  $S - P$  is an Ore set, there exist  $t_1 \in S$  and  $s_1 \in S - P$  such that  $s_1 t = t_1 s$ . From the above we can conclude that  $xy^{-1} = ts^{-1} = s_1^{-1} t_1$  and  $s_1 x = t_1 y$ . Thus  $y$  is a left-divisor of  $s_1 x$ . Since  $l - \gcd\{x, y\} = 1$ , by Corollary 2.5,  $y$  is a left-divisor  $s_1$ . Therefore,  $s_1 = ky$  for some  $k \in S$  and  $s_1 \in P$ , a contradiction. By a similar reasoning it follows that  $yx^{-1} \notin S_P$ . Thus  $S_P$  is not a total valuation ring.  $\square$

**Corollary 2.8.** Let  $S$  be a  $\nu$ -Bezout order in a division ring  $D$  such that  $S$  is localizable at every completely prime ideal. Then:

- (i) Every completely prime ideal contained in a PF-prime ideal is again a PF-prime ideal.
- (ii) The set of all completely prime ideals contained in a PF-prime ideal is linearly ordered, and hence the set of all PF-prime ideals forms a tree.

### 3. STAR OPERATION ON ORDERS IN A SIMPLE ARTINIAN RINGS

Throughout of the rest of the paper  $S$  is an order in a simple Artinian ring  $Q$  and  $F_r(S)$  ( $\bar{F}_r(S)$ ) are the set of nonzero right  $S$ -ideals ( $S$ -submodules) of  $Q$ .

**Definition 3.1.** A mapping  $I \rightarrow I^*$  of  $\bar{F}_r(S)$  into  $\bar{F}_r(S)$  is called a semistar operation on  $S$  if the following three conditions hold for all  $u \in U(Q)$  and  $I, J \in \bar{F}_r(S)$ :

- (1)  $(uI)^* = uI^*$ ;
- (2) if  $I \subseteq J$  then  $I^* \subseteq J^*$ ;
- (3)  $I \subseteq I^*$  and  $(I^*)^* = I^*$ .

When  $S^* = S$  the restriction  $*$  to  $F_r(S)$  also satisfies to the above tree conditions and is called a star operation on  $S$ . An element  $I \in F_r(S)$  is called a right star ideal if  $I = I^*$ .

**Example 3.2.** (i)  $i_d : \bar{F}_r(S) \rightarrow \bar{F}_r(S)$  defined by  $I^{i_d} = I$ , the identity map on  $\bar{F}_r(S)$ , is a semistar operation on  $S$ .

(ii) The map  $\nu : \bar{F}_r(S) \rightarrow \bar{F}_r(S)$  defined by  $I^\nu := (S : (S : I)_l)_r$  is a semistar operation.

(iii) If  $*$  is a semistar operation on  $S$  then we can define the map  $*_f : \bar{F}_r(S) \rightarrow \bar{F}_r(S)$  by  $I^{*_f} := \cup F^*, F \in \bar{F}_r(S)$  where the union runs over all finitely generated  $F \subseteq I$  such that  $F \in \bar{F}_r(S)$ . It is easy to see that  $*_f$ , which is called a semistar operation of finite type associated to  $*$ , is indeed a semistar operation. In particular, the semistar operation of finite type associated to  $\nu$  is denoted by  $\tau$ .

**Lemma 3.3.** *Let  $*$  be a star operation on  $S$  and  $\{F_\alpha\}$  a family of elements of  $F_r(S)$ . Then  $(\sum_\alpha F_\alpha)^* = (\sum_\alpha F_\alpha^*)^*$ .*

*Proof.* The proof is the same as in the commutative case, see [13, Proposition 32.2].  $\square$

**Lemma 3.4.** *Let  $S$  be an order in a simple Artinian ring  $Q$  and  $A, B$  right  $S$ -ideals. Then:*

- (1) *If  $A^\tau = B^\tau$  then  $A^\nu = B^\nu$ .*
- (2) *If  $A$  is a  $\nu$ -ideal, then  $A$  is a  $\tau$ -ideal.*
- (3) *If  $S$  satisfies the ascending chain condition on the set of left  $\nu$ -ideals, then every right  $\tau$ -ideal is a right  $\nu$ -ideal.*
- (4)  *$S$  satisfies the ascending chain condition on the set of left  $\nu$ -ideals if and only if it satisfies this condition on the set of left  $\tau$ -ideals.*

*Proof.* (1) This follows from  $A \subseteq A^\tau \subseteq A^\nu$  and  $(A^\nu)^\nu = A^\nu$ .  
 (2) Since  $A$  is a  $\nu$ -ideal and  $A^\tau \subseteq A^\nu$ , we have  $A^\tau = A$ .  
 (3) Let  $A$  be a right ideal and  $\{B_i\}$  the family of right ideals of  $S$  contained in  $A$ . Then  $(S : A)_l \subseteq (S : B_i)_l$  for all  $i$ . Since each  $(S : B_i)_l$  is a left  $\nu$ -ideal and  $S$  satisfies the ascending chain condition on the set of all left  $\nu$ -ideals, there exists a minimal element in  $\{(S : B_i)_l\}$ , say  $(S : B_n)_l$ . If  $B_n^\nu \neq A$ , there exists an element  $b \in A - B_n^\nu$ . Let  $B = B_n + bS$ . Then  $B$  is a finitely generated right ideal and  $B \subseteq A$  such that  $(S : B)_l \subset (S : B_n)_l$ . This contradicts the minimality of  $(S : B_n)_l$  so that  $B_n^\nu = A$ . Now let  $A$  be a right  $\tau$ -ideal. By the above there exists a finitely generated right  $S$ -ideal  $B$  such that  $B \subseteq A$  and  $B^\nu = A^\nu$ . Since  $B^\tau = B^\nu$  and  $A$  is right  $\tau$ -ideal, we conclude that  $A \subseteq A^\nu \subseteq B^\nu = B^\tau \subseteq A^\tau = A$ . Thus  $A^\nu = A$ .  
 (4) This follows from (3) and (4).  $\square$

**Lemma 3.5.** *Let  $A_1 \subset A_2 \subset \dots$  be right  $\tau$ -ideals. Then  $A = \cup_{i \geq 1} A_i$  is a right  $\tau$ -ideal.*

*Proof.* Let  $B$  be a finitely generated right  $S$ -ideal with  $B \subset A$ . Then there exists  $A_n$  such that  $B \subseteq A_n$ . Thus  $B^\nu \subseteq A_n^\tau = A_n$ , so that  $B^\nu \subseteq A$ .  $\square$

**Corollary 3.6.** *If every right  $\tau$ -ideal is a finitely generated  $S$ -ideal, then  $S$  satisfies the ascending chain condition on the set of all right  $\tau$ -ideals.*

*Proof.* If  $S$  does not satisfy the ascending chain condition on the set of all right  $\tau$ -ideals, then there exists an infinite chain of right  $\tau$ -ideals,  $A_1 \subset A_2 \subset \dots$ . By Lemma 3.5,  $A = \cup_{i \geq 1} A_i$  is a  $\tau$ -ideal which is not a finitely generated, a contradiction.  $\square$

**Lemma 3.7.** *Let  $V$  be a total valuation ring of rank of least 2 in a division ring  $K$ , and  $Q$  a non-maximal completely prime ideal of  $V$ . Put  $T = V + V_Q[x, \sigma]x$ . Then  $M = \{f \in T : f(0) \notin U(V)\}$  is a  $\tau$ -ideal.*

*Proof.* First we show that  $M$  is a completely prime ideal of  $T$ . Let  $f, g \in M$ . Since  $V$  is a total valuation ring, we can assume that  $f(0) = g(0)s$  for some  $s \in V$ . Now if  $(f(0) - g(0))u = 1$  for some  $u \in V$  then

$$g(0)(s - 1)u = (g(0)s - g(0))u = (f(0) - g(0))u = 1,$$

which shows that  $g(0) \notin M$ , a contradiction. It is clear that  $ft, tf \in M$  for all  $f \in M$  and  $t \in T$ . Thus  $M$  is an ideal of  $T$ . If  $f, g \in M$ , then there exists  $u, v \in V$  such that  $ug(0) = g(0)u = 1$  and  $vf(0) = f(0)v = 1$ . Now we have  $(fg)(0)uv = f(0)g(0)uv = f(0)v = 1$ , so that  $fg \notin M$ , and  $M$  is a completely prime ideal.

Next we show that  $M$  is a  $\tau$ -ideal. We note that  $J(V) \subseteq M$ , and so  $M \cap (V - Q) \neq \emptyset$ . It is easy to show that  $M = J(V) + V[x, \sigma]x$ . Now let  $F$  be a finitely generated right  $R$ -ideal such that  $F \subseteq M$ . Then there exist  $f_1, \dots, f_n \in M - 0$  with  $F = f_1T + \dots + f_nT$ . Since  $f_i(0) \in J(V)$  and the set of all right ideal of  $V$  is totally ordered, without loss of generality we can write  $f_1(0)V \subseteq f_2(0)V \subseteq \dots \subseteq f_n(0)V$ . Thus  $f_n(0)$  a right divisor of  $f_i(0)$  for all  $i$ . Now if  $f_n(0) \in Q$ , then every  $s \in J(V) - Q$  is a right divisor of  $f_n(0)$ . Hence, there exists an element  $s \in J(V) - Q$  such that  $s$  is a right divisor  $f_i(0)$  for all  $i$ . Since  $x = ss^{-1}x$ , the element  $s$  is a right divisor of  $f_i$  for all  $i$ . Thus  $f_1T + \dots + f_nT \subseteq sT$  and  $(f_1T + \dots + f_nT)_\nu \subseteq sT \subseteq M$ , which implies that  $M$  is a right  $\tau$ -ideal.

In much the same way one can show that  $M$  is a left  $\tau$ -ideal.  $\square$

**Remark 3.8.** If  $*$  :  $F_r(S) \rightarrow F_r(S)$  is a star operation on  $S$ , and if  $I_r(S)$  is the set of all integral right ideals of  $S$ , then we have  $I \subseteq I^* \subseteq S^* = S$  for all  $I \in I_r(S)$ . Therefore, each star operation on  $S$  induces a function  $I \rightarrow I^*$  on  $I_r(S)$  such that conditions (1)–(3) of Definition 3.1 are satisfied. Moreover, if  $F \in F_r(S)$ , then  $F = q^{-1}I$  for some  $I \in I_r(S)$ , and  $F^* = (q^{-1}I)^* = q^{-1}I^*$ . Therefore, a star  $*$  on  $S$  is completely determined by its action on  $I_r(S)$ .

In the following lemma  $F_r^*(S)$  denotes the set of all right  $*$ -ideals of  $S$ .

**Lemma 3.9.** A nonempty subset  $F' \subset F_r(S)$  satisfies  $F' = F_r^*(S)$  for some  $*$ -operation on  $S$  if and only if all of the following conditions are satisfied:

- (1)  $S \in F'$ ;
- (2)  $I \in F'$  implies that  $uI \in F'$  for each  $u \in U(Q)$ ;
- (3)  $\emptyset \neq \{I_\alpha\} \subseteq F'$  with  $\cap I_\alpha \neq \{0\}$  implies that  $\cap I_\alpha \in F'$ .

If (1)–(3) hold we can define  $I^* = \cap\{J \in F' : I \subseteq J\}$ .

*Proof.* Suppose that  $*$  is a star operation on  $S$ . By Definition 3.1 and part (b) of [13, Proposition 32.2], the set  $F_r^*(S)$  satisfies (1)–(3). We always have  $I^* = \cap\{J \in F_r^*(S) : I \subseteq J\}$ . Conversely, for  $I \in F_r(S)$  we define  $I^* := \cap\{J \in F' : I \subseteq J\}$ . Since the intersection of any collection of right  $S$ -ideals is again a right  $S$ -ideal, we have  $I^* \in F_r(S)$ . From (1) and (2) we can conclude that  $S^* = S$ , and  $(uI)^* = uI^*$  for all  $I \in F_r(S)$ , and  $u \in U(Q)$ . The definition of  $*$  implies that  $I \subseteq I^*$  and  $I^* \subseteq J^*$ , wherever  $I \subseteq J$ . Now it is clear from the definition and condition (3) that  $F' = \{I \in F_r(S) : I = I^*\}$ . Thus  $(I^*)^* = I^*$ .  $\square$

**Definition 3.10.** A non-Artinian ring  $S$  in a simple Artinian ring  $Q$  is called a discrete Dubrovin valuation ring if  $S$  is a maximal Dubrovin valuation ring of  $Q$  and  $J(S) \neq J(S)^2$ .

A Dubrovin valuation ring  $S$  is discrete if and only if  $S \cap F$  is a discrete valuation ring of the field  $F$ , where  $F$  is the center of  $Q$ , see part c of [19, Proposition 2.7]. There are several equivalent definitions to Definition 3.10, see [19, Theorem 2.6].

**Lemma 3.11.** Let  $S$  be a discrete Dubrovin valuation ring of a simple Artinian ring  $Q$ . Then  $*$  =  $i_d$  for all nontrivial star operations  $*$  on  $S$ .

*Proof.* Since  $I \subseteq I^* \subseteq I_\nu$ , it is enough to prove that every right  $S$ -ideal is divisorial. Since  $S$  has rank one, every principal right  $S$ -ideal is a two sided  $S$ -ideal. That is,  $aS = Sa$  for all nonzero  $a \in Q$  (see the discussion before [7, Lemma 8]). Let  $H_r(S)$  and  $H(S)$  be the set of all principal right  $S$ -ideals and principal  $S$ -ideals respectively. Then by part (i) of [7, Theorem 9],  $H_r(S) = H(S) \cong F_r(S) = F(S)$ , which show that every right  $S$ -ideal is divisorial.  $\square$

**Lemma 3.12.** *Let  $S$  be a Dubrovin valuation ring of  $Q$  such that  $*$  =  $i_d$  for all nontrivial semistar operations  $*$  on  $S$ . Then  $S$  is discrete.*

*Proof.* Assume that  $J(S) = J(S)^2$ . Then, by [18, Proposition 1.3], for every  $a \in S$  the right ideal  $aJ(S)$  is not divisorial. Thus  $\nu \neq i_d$ , which is a contradiction. Therefore,  $J(S) \neq J(S)^2$ . Now let  $T$  be a proper overring of  $S$  in  $Q$ . We define  $I^{*T} = IT$  for every  $S$ -submodule  $I$  of  $Q$ . It is easy to see that  $*_T$  is a semistar on  $S$ , and  $I^{*T} = \cup\{JT : J \subseteq I, J \text{ is finitely generated}\}$ . Thus  $*_T$  is semistar and  $S^{*T} = ST = T \neq S$ . This is a contradiction, because  $*_T \neq i_d$ . Hence  $S$  is a maximal Dubrovin valuation ring.  $\square$

#### 4. RIGHT PRÜFER $\nu$ -MULTIPLICATION ORDERS

An integral domain in which every finitely generated ideal is invertible is called a Prüfer domain. An integral domain in which every finitely generated ideal is  $t$ -invertible is called a Prüfer  $\nu$ -multiplication domain and denoted by  $P\nu MD$ . The  $P\nu MD$ s include Krull domains, Prüfer domains, GCD-domains and unique factorization domains. The aim of this section is introduce a non-commutative version of  $P\nu MD$ s in simple Artinian rings. This version includes  $P\nu MD$  right Bezout orders, right GCD-orders and right Prüfer orders.

For an additive subgroup  $I$  of a simple Artinian ring  $Q$  we define  $O_l(I) := \{q \in Q : qI \subseteq I\}$ ,  $O_r(I) := \{q \in Q : Iq \subseteq I\}$  and  $I^{-1} := \{q \in Q : IqI \subseteq I\}$ .

Recall that an order  $S$  in a simple Artinian ring  $Q$  is called a right Prüfer ring if any finitely generated right  $S$ -ideal  $I$  is left invertible as a right  $S$ -ideal and right invertible as a left  $O_l(I)$ -ideal. More precisely we have:

**Definition 4.1.** *An order  $S$  in a simple Artinian ring  $Q$  is a right Prüfer order if and only if  $I^{-1}I = S$  and  $II^{-1} = O_l(I)$  for any finitely generated right  $S$ -ideal  $I$ .*

A left Prüfer order is defined similarly. A Prüfer order  $S$  is a right as well as a left Prüfer order. By [24, Lemma 1.4 and 1.5], an order  $S$  in a simple Artinian ring  $Q$  is a right Prüfer order if and only if  $(S : I)_l I = S$  and  $I(S : I)_l = O_l(I)$  for any finitely generated right  $S$ -ideal  $I$ .

**Lemma 4.2.** *Let  $S$  be a Prüfer order in a simple Artinian ring  $Q$  with finite dimension over its center. For any nonzero  $I \in F_r(S)$  we define  $I \rightarrow I^\omega = \cap IS_M$ , where  $M$  runs over all maximal ideals of  $S$ . Then:*

- (i)  $\omega$  is a star operation and  $IS_M = I^\omega S_M$ .
- (ii) If  $I$  is a finitely generated right ideal of  $S$  such that  $(JI)^\omega \subseteq (KI)^\omega$  and rank  $S_M$  is one for every  $M$ , then  $J^\omega \subseteq K^\omega$ .
- (iii) If  $S$  is Noetherian then  $\omega = \nu$ .

*Proof.* (i) By [24, Theorem 22.8] and [26, Lemma 2.4], for each maximal ideal  $M$  of  $S$  the localization  $S$  at  $M$  exists and  $R_M$  is a Dubrovin valuation ring. Furthermore,  $S = \cap S_M$  where  $M$  runs over all maximal ideals of  $S$ . First

we show that  $I^\omega \in F_r(S)$  for all  $I \in F_r(S)$ . Then we prove that  $\omega$  satisfies all three conditions (1)–(3) of Definition 3.1. Since  $I^\omega S = (\cap_M IS_M)S = \cap_M IS_M S = \cap_M IS_M = I^\omega$ , the  $I^\omega$  is a nonzero right  $S$ -submodule of  $Q$ . Now let  $u \in U(Q)$  such that  $uI \subseteq S$ . Then  $uI^\omega = u(\cap_M IS_M) = \cap_M uIS_M \subseteq \cap_M SS_M = \cap_M S_M = S$ . Therefore,  $I^\omega \in F_r(S)$ . To prove that  $I \rightarrow I^\omega$  is a star operation, we need to show that  $\omega$  satisfies all three conditions (1)–(3) of Definition 3.1.

To prove condition of (1), let  $u \in U(Q)$ . Then  $uI^\omega = u \cap_M IS_M = \cap_M uIS_M = \cap_M (uI)S_M = (uI)^\omega$ , which implies (1).

That condition (2) holds is clear.

To prove condition (3) we can write  $I = u^{-1}J$ , where  $J$  is a right integral  $S$ -ideal and  $u \in U(Q)$ . By part (1) we have  $I^\omega = u^{-1}J^\omega$  and  $(I^\omega)^\omega = u^{-1}(J^\omega)^\omega$ . Therefore, we only need to show that  $J^\omega = (J^\omega)^\omega$ . Since the right ideal  $JS_M$  is an extension of the right ideal  $J$  of  $S$  to  $S_M$  and  $JS_M \subseteq S_M$ , we have  $JS_M = (JS_M \cap S)S_M = (JS_M \cap (\cap_M S_M))S_M \supseteq (\cap_M JS_M)S_M \supseteq J^\omega S_M$ . Thus  $J^\omega = \cap_M JS_M \supseteq \cap_M J^\omega S_M = (J^\omega)^\omega$ . It follows that  $I \rightarrow I^\omega$  is a  $\omega$ -operation. From the fact that  $JS_M \supseteq J^\omega S_M$ , we conclude that  $IS_M = I^\omega S_M$ .

- (ii) From (i) and  $(JI)^\omega \subseteq (KI)^\omega$ , we have  $(JI)S_M = (JI)^\omega S_M \subseteq (KI)^\omega S_M = (KI)S_M$ . From the fact that  $S_M$  is a Dubrovin valuation ring,  $IS_M$  is a finitely generated right  $S_M$ -ideal and using [24, Corollary 5.5], we have  $IS_M = qS_M$  for some regular element  $q \in Q$ . Since  $S_M$  is a rank one and  $q$  is a unit in  $Q$ , we have  $qS_M q^{-1} = S_M$ . Thus  $(JI)S_M = JqS_M = (JS_M)q \subseteq (KI)S_M = KqS_M = (KS_M)q$ , and so  $(JS_M) \subseteq (KS_M)$ . Hence  $J^\omega \subseteq K^\omega$ .  $\square$

Let  $S$  be an order in a simple Artinian ring  $Q$  and  $*$  is a star operation on  $S$ . A right  $*$ -ideal  $I$  is called of *finite type* if there exists a finitely generated right  $S$ -ideal  $J$  such that  $I = J^*$ . Let  $* = \nu$  and  $H_r(R)$  be the set of all right  $\nu$ -ideals of finite type.

**Definition 4.3.** An order  $S$  in a simple Artinian ring  $Q$  is called a right Prüfer  $\nu$ -multiplication order if

$$((S : I)_l I)^\tau = S \quad \text{and} \quad (I(S : I)_l)^\tau = O_l(I)$$

for every  $I \in H_r(R)$ .

A left Prüfer  $\nu$ -multiplication order is defined similarly. A Prüfer  $\nu$ -multiplication order is simultaneously a right and a left  $\nu$ -Prüfer order.

**Lemma 4.4.** The following are right Prüfer  $\nu$ -multiplication orders:

- (1) Commutative Krull domains;
- (2) Commutative Prüfer  $\nu$ -multiplication domains;
- (3) Right Bezout orders;
- (4) Right  $\nu$ -Bezout orders;
- (5) Right Prüfer orders;

*Proof.* Every commutative Krull domain is a commutative Prüfer  $\nu$ -multiplication domain. To prove (1) and (2) it is thus enough to show that the Prüfer  $\nu$ -multiplication domain  $S$  is a right Prüfer  $\nu$ -multiplication order. To see this, let  $I$  be a right  $S$ -ideal of finite type. From the fact that  $S$  is commutative and  $I$  has



an inverse with respect to  $\tau$  multiplication, we have  $I[S : I]_l = S = [S : I]_l I$  and  $O_l(I) = S$ . Thus  $S$  is a right  $\nu$ -multiplication order in the fraction field  $F$  of  $S$ .

To prove (3) we only need to prove (4), because every right Bezout order is a right  $\nu$ -Bezout order. To establish (4), let  $S$  be a right  $\nu$ -Bezout order and  $I$  be a right  $S$ -ideal of finite type. Then  $I = J^\nu$  for some finitely generated right  $S$ -ideal  $J$ . Since  $S$  is right  $\nu$ -Bezout we have  $I = J^\nu = qS$  for some regular element  $q \in Q$ . It is easy to show that  $[S : I]_l = [S : qS]_l = Sq^{-1}$  and  $O_l(I) = qSq^{-1}$ . Therefore,  $I^{-1}I = Sq^{-1}qS = S$  and  $I^{-1}I = (qS)(Sq^{-1}) = qSq^{-1} = O_l(I)$ , as desired.

Proving (5), the set of all finitely generated right  $S$ -ideals coincides with  $H_r(S)$ . Now let  $I \in H_r(S)$ . Then  $[S : I]_l I = S$  and  $I[S : I]_l = O_l(I)$ . Hence  $([S : I]_l I)^\tau = S$  and  $(I[S : I]_l)^\tau = O_l(I)$ .  $\square$

In [10, Theorem 1] Dubrovin proved that every non-commutative Prüfer order is a semi-hereditary order. The following example taken from [25], shows that the converse is not necessarily true.

**Example 4.5.** Let  $p$  be an odd prime and  $D = \mathbb{Z}_p[t]_{t\mathbb{Z}_p[t]}$ . Then  $D$  is a local ring with maximal ideal  $m = tD$ . Let  $F$  be the quotient field of  $\mathbb{Z}_p[t]$  which is also the quotient field of  $D$ . We define the automorphism  $\sigma$  on  $\mathbb{Z}_p[t]$  by  $t^\sigma = -t$  and  $a^\sigma = a$  for all  $a \in \mathbb{Z}_p$ . Then  $\sigma$  can naturally be extended to  $D$ . Now let  $S = D[x]_{xD[x]}$ . Then  $S = \{f(x)g(x)^{-1} : f(x), g(x) \in D[x] \text{ with } g(0) \neq 0\}$ . Define the epimorphism  $\Phi$  from  $S$  to  $F$  by  $\Phi(f(x)g(x)^{-1}) = f(0)g(0)^{-1}$ , and let  $R = \Phi^{-1}(D)$ . Then  $R$  is a valuation ring of rank 2 and  $P_0 + mR, P_0$  and  $(0)$  are the only prime ideals of  $R$ . The automorphism  $\sigma$  is extended to an automorphism of  $D[x]$  by  $(f(x))^\sigma = a_n^\sigma x^n + \dots + a_0^\sigma$  for any  $f(x) = a_n x^n + \dots + a_0 \in D[x]$ . The group  $G$  generated by  $\sigma$  has order of 2 and the skew group ring  $R \star G$  is a semihereditary order which is not Prüfer.

**Question 4.6.** Is  $R \star G$  a right Prüfer  $\nu$ -multiplication order?

Recall that a ring  $R$  is called a generalised discrete valuation ring, if the set of right ideals are inversely well-ordered by inclusion. All right ideals of such a ring are two-sided and actually principal right ideal. Now we give the definition of non-commutative Krull domain in the sense of Brungs.

**Definition 4.7.** An integral domain  $R$  is called a non-commutative Krull domain if there is a family of generalised discrete valuation domains  $V_i, i \in I$ , satisfying the following:

- (1)  $R = \cap_{i \in I} V_i$  and each  $V_i$  is a subring of the same division ring  $Q$  such that  $Q = Q(V_i)$  for every  $i \in I$ .
- (2) Every  $a \in R$  is a unit for all but a finite number of  $V_i$ .
- (3) Each  $V_i$  satisfies as  $R_{P_i}$  for prime ideal  $P_i$  of  $R$  such that  $P_i \cap P_j$  contains no nonzero prime ideal for  $i \neq j$ .

Every one-sided ideal of  $R$  is two-sided and  $aR = \cap_{i \in I} aV_i$  for all  $a \in Q^*$ . Thus  $F_r(R) = F_l(R) = F(R)$ . We define the operation  $b$  by  $A^b = \cap AV_i$ , where  $A \in F_r(R)$ . It is easy to show that  $b$  is a star operation. The equivalence relation on  $F_r(R)$  can be defined by  $A \sim B$  whenever  $A^b = B^b$ . For each  $A \in F_r(R)$ , we denote by  $[A]$  the equivalence class determined by  $A$ . Let  $D(R)$  be the set of all equivalence classes. Then by [20, Theorem 2.3],  $D(R)$  is an abelian group with defined operation  $[A][B] = [A^b B^b]$ .  $A^b \subseteq A^\tau$  for all  $A \in F_r(R)$ . Hence non-commutative Krull domains are examples of non-commutative Prüfer  $\nu$ -multiplication domains.

**Example 4.8.** Let  $F$  be a field of characteristic zero and  $A := F[x, y]$  with  $xy - yx = 1$ . Then  $A[z]$  is an Asano order in  $Q(A[z])$  [17]. For every non unit  $a \in A$  the right ideal  $I := aA[z] + zA[z]$  generated by  $a, z$  is not projective. Thus  $A[z]$  is not a right semi hereditary order and hence not a right Prüfer order. On the other hand,  $A[z]$  is a Krull domain in the sense of Brungs and hence a Prüfer  $\nu$ -multiplication order.

Recall that two orders  $R$  and  $S$  in a simple Artinian ring  $Q$  is called equivalent if there exist regular elements  $a_1, a_2, b_1, b_2 \in Q$  such that  $a_1 R b_1 \subseteq S$  and  $a_2 S b_2 \subseteq R$ .

**Lemma 4.9.** Let  $S$  be a bounded Krull ring in the sense of Marubayashi. Then  $O_l(I)$  is a bounded Krull ring for any divisorial right  $S$ -ideal  $I$ .

*Proof.* By [21, Lemma 2.5]  $O_l(I)$  is a maximal order equivalent to  $S$ , and by [21, Theorem 2.6]  $O_l(I)$  is a bounded Krull ring.  $\square$

**Lemma 4.10.** Let  $S$  be a Noetherian bounded Krull ring in the sense of Marubayashi. Then  $S$  is a right Prüfer  $\nu$ -multiplication order.

*Proof.* By [21, Corollary 1.4]  $S$  is a maximal order in the sense of Asano. Now let  $I$  be right  $S$ -ideal of finite type. Then, by part 4 of [22, Lemma2],  $((S : I)_l I^\nu)^\nu = S$  and  $(I^\nu(S : I)_l)^\nu = O_l(I)$ . The ring  $S$  is Noetherian and  $O_l(I)$  is equivalent to  $S$ . Hence  $O_l(I)$  is also Noetherian. Therefore,  $(S : I)_l I^\nu$  and  $I^\nu(S : I)_l$  are finitely generated right ideals of  $S$  and  $O_l(I)$  respectively. Thus  $((S : I)_l I)^\tau = ((S : I)_l I^\nu)^\nu = S$  and  $(I(S : I)_l)^\tau = (I^\nu(S : I)_l)^\nu = O_l(I)$ , as desired.  $\square$

The following is a generalisation of [24, Proposition 2.6] to right Prüfer  $\nu$ -multiplication orders.

**Proposition 4.11.** Let  $S$  be a right Prüfer  $\nu$ -multiplication order in simple Artinian ring  $Q$  and  $T$  is an overring of  $S$ . Then  $T$  is also a Prüfer  $\nu$ -multiplication order in  $Q$ .

*Proof.* Let  $I' \in H_r(T)$ . Then  $I' = (a_1 T + \cdots + a_n T)^\nu$  for some finitely generated right  $T$ -ideal of  $a_1 T + \cdots + a_n T$ . Put  $I = (a_1 S + \cdots + a_n S)^\nu$ . Then  $I \in H_r(S)$  and  $((S : I)_l I)^\tau = S$  and  $(I(S : I)_l)^\tau = O_l(I)$ . Now let  $x \in (S : I)_l$ . Then  $x(a_1 S + \cdots + a_n S) \subseteq xI \subseteq S$  and so  $x(a_1 S + \cdots + a_n S)T \subseteq xIT \subseteq ST$ . Thus  $x(a_1 T + \cdots + a_n T) \subseteq T$  and  $xI' \subseteq T$ , which shows that  $(S : I)_l \subseteq (T : I')_l$ . From  $(S : I)I \subseteq (T : I')_l I' \subseteq T$ ,  $I(S : I) \subseteq I'(T : I')_l \subseteq O_l(I')$  and since  $S$  is a right Prüfer  $\nu$ -multiplication order, we have  $S = ((S : I)I)^\tau \subseteq ((T : I')_l I')^\tau \subseteq T$  and  $O_l(I) = (I(S : I))^\tau \subseteq (I'(T : I')_l)^\tau \subseteq O_l(I')$ . Hence  $1 \in ((T : I')_l I')^\tau$  and  $1 \in (I'(T : I')_l)^\tau$ . Therefore,  $((T : I')_l I')^\tau = T$  and  $(I'(T : I')_l)^\tau = O_l(I')$ .  $\square$

**Remark 4.12.** By [24, Lemma 1.5] a right  $S$ -ideal  $I$  of  $Q$  is a projective  $S$ -module if and only if  $I(S : I)_l = O_l(I)$ . Thus if  $I$  is projective  $S$ -module, we always have  $(I(S : I)_l)^\tau = O_l(I)$  but the converse is not true. For example, the Krull ring  $S = \mathbb{F}[x, y]$  is not a semi-hereditary. Thus there exists a finitely generated  $S$ -ideal  $I$  which is not projective. Now, since  $S$  is a Prüfer  $\nu$ -multiplication domain, we have  $(I(S : I)_l)^\tau = O_l(I)$ . In [24, Proposition 2.5] it is proven that a right Prüfer order is left Prüfer and vice versa. The proof relies on the semi-hereditariness of Prüfer orders and [24, Lemma 1.5]. The Prüfer  $\nu$ -multiplication order is not necessarily semi-hereditary. As yet we have been unable to prove or disprove that a right Prüfer  $\nu$ -multiplication order is a left  $\nu$ -multiplication order.

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